The Effect of Product Differentiation on Cartel Stability
– Explaining Partial Cartels

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Abstract

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Cartels may cause great harm to consumers and economic efficiency. However, literature on endogenous cartel formation with dynamic competition is scarce. This paper is the first to endogenize cartel formation in a model with differentiated products. In a model with symmetric firms, linear costs and quadratic profit functions, I find that the size of a stable cartel decreases (increases) when products become more differentiated (homogeneous). Furthermore, I find that the size of the smallest stable cartel rarely exceeds six to seven firms, as the relative pay-off from free-riding increases faster in cartel size than the pay-off from collusion – irrespective of industry size. Finally, I find that industry-wide cartels may not be incentive compatible when products are sufficiently homogeneous. Otherwise, the incentive compatibility constraint likely holds. Note that this draft contains numerical solutions to the cartel-formation stage, as analytical results remain a work in progress.

Keywords Collusion, Cartel formation, Cartel stability, Differentiated products

JEL Classification D43, L11, L41

1 Introduction

Real-life cartels – more often than not – include only some of the firms in a market. Hay & Kelley (1974) report statistics such as market shares and industry size (number of firms) for

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over 60 US cartels in the period 1963-1972. Their overview shows that 19 of 31 cartels (for which market shares are reported) are partial, in the sense that they do not include all firms in the market. The overview also shows that these cartels on average have a market share of 92 percent (share of sales) and make up 74 percent of firms in an industry. 53 of 54 cartels studied in Griffin (1989) were partial. More recent examples from European Commission decisions between 2000-2004 include the 1990s citric acid cartel (that encompassed 60 percent of global production), cartels in vitamins B1, B2 and C (that excluded Chinese suppliers), and the European industrial tubes cartel (that lasted 13 years, excluded two significant suppliers, and controlled 75-85 percent of total production) (Harrington 2006). Observations of actual cartels therefore suggest that an assumption of all-inclusive cartels is unrealistic. This raises an important theoretical question: What firm or product characteristics determine cartel participation and size? A model of quantity competition and a model of differentiated price competition may give different answers to that question.

In reality, markets rarely consist of perfect substitutes, but rather of a limited number of differentiated firms or brands. Furthermore, consumers are likely to prefer one brand or product to another, and may differ in their preferences. This is commonly referred to as horizontal product differentiation. Intuitively, firms gain more by colluding with a close competitor than they do by colluding with weaker substitutes. Collusive agreements including all firms in a market may therefore be less desirable for the firms. As most mergers take place in industries with differentiated products, formulating a model for cartel formation in differentiated goods markets may be of particular importance for antitrust work.

For example, incentives for collusion, cartel stability and cartel formation may differ in markets with homogeneous and differentiated products. If competition authorities are interested in which mergers give rise to coordinated effects, the answer may differ depending on what we assume about the mode of competition. Bos & Harrington (2010) show that mergers between intermediate-size firms may cause the most severe coordinated effects when firms are capacity constrained (with homogeneous products). However, if firms instead engage in price competition with differentiated products, firm size may be less important.\(^1\)

As in a capacity-constrained price game, a model of price competition between firms with differentiated products introduces a free-riding incentive: the existence of a (partial) cartel in a market raises demand for non-colluding firms’ products. Therefore, firms may have incentives to free ride rather than part-take in collusion and an incentive compatible cartel may only include a subset of firms in a given market. Thus, modelling collusion in markets

\(^1\)Note that there exist models of coalitions in markets with differentiated price competition in the merger literature. For example Deneckere & Davidson (1985) present a model of coalitions with Bertrand competition. The article shows that profits are increasing in the size of a coalition, and that free-riders benefit more than coalition members. However, the article does not assess coalition stability and treats coalition size(s) as exogenously given.
with differentiated products may be important to obtain a more realistic model of real-world markets – and offer an alternative explanation of why cartels typically do not include all firms. Using a model of symmetric firms, engaging in differentiated price competition, this article addresses the following research questions: (i) Under what industry conditions can we expect a cartel to be industry-wide?, and (ii) What market characteristics matter the most for the degree of cartelization? I find that the market-wide cartel is stable if products are sufficiently homogeneous and the industry is sufficiently small. Furthermore, the size of a stable partial cartel is increasing in product homogeneity, but does not exceed seven firms – irrespective of industry size. Cartel size of partial cartels is limited as the relative pay-off from free-riding increases faster in cartel size than the pay-off from collusion.

The literature on endogenous cartel formation originates from static models. In these models, cartel formation is characterized as a two-stage game. Early works include Selten (1973), d’Aspremont et al. (1983), Donsimoni (1985), and Donsimoni et al. (1986). In the literature on collusion, it is common to assume quantity competition. See for example Yi (1997) and Compte et al. (2002). These coalition models generally consist of a two-stage coalition game, where firms first choose coalition membership. Subsequently, they either play a game of quantity competition or a capacity constrained price game. These models assume that products are homogeneous and therefore perfectly substitutable. As competition is modelled as a one-shot game, the cartel members must be able to enter into a binding agreement on pricing or quantity choice.

So far, little work has been done on optimal cartel size and partial cartelization of industries with dynamic competition. Escrihuela-Villar (2008, 2009), and Bos & Harrington (2010) do however present models where the cartel formation stage is followed by dynamic competition models (infinitely repeated games). All three articles assume homogeneous products. The articles model games of capacity-constrained price competition or quantity competition. For markets with differentiated products, Deneckere & Davidson (1985) shows that mergers are increasingly profitable in the (exogenous) number of merging firms, and that non-merging firms gain more from the merger than the merging firms. A merger in the context of Deneckere & Davidson (1985) is equivalent to a cartel that maximizes joint profit. The article shows that, contrary to quantity competition, mergers with differentiated price competition are advantageous for the merging firms. However, the article treats merger size as exogenous and does not assess stability of the agreement. This article will, to my knowledge, be the first to examine endogenous cartel formation under the assumption of price competition with differentiated products.

In this paper, I introduce a simple model with no renegotiation and a grim-trigger punishment
strategy. The model may serve as a baseline for assessing endogenous cartel formation with differentiated products. In section 2, I will present the basic set-up of the model. Furthermore, I present my modelling choices for the consumer and firm side of the industry. Section 3 presents general equilibrium pricing strategies for firms when there exists a cartel in the industry. Finally, I assess whether the incentive compatibility constraint holds. I find that if firms are sufficiently patient, collusion can be sustained for any partial cartel, and that the incentive compatibility constraint rarely binds. In section 4, I present numerical results for the cartel formation stage. I show that full cartelization rarely is optimal, as the condition for internal cartel stability only is satisfied when products are relatively homogeneous. Furthermore, I find that closeness in competition between firms is the most important parameter to determining stable cartel size. Section 5 concludes and summarizes remaining work with this model.

2 The model

Figure 1 illustrates the timing of the game. In the first period, firms simultaneously choose whether or not to become a cartel member. Firms only make the choice of cartel-membership once. This implies that renegotiation and recartelization is not possible. Clearly, no renegotiation may be an unrealistic assumption. However, the assumption simplifies the model and eases the analysis of comparative statics, allowing me to focus on optimal cartel size.

After the cartel-membership choice, firms play an infinitely repeated price-game with differentiated products. In each stage-game of the price-game, firms simultaneously choose prices. In this price game, cartel members will choose prices differently than non-members. I will assume that cartel break-down will lead to a reversion to the Nash equilibrium of the stage game for all future periods (the punishment strategy is a so-called trigger strategy).

![Figure 1: Timing of the game](image.png)

2.1 Modelling consumers

To ensure an analytical solution to the problem, I assume that demand takes a linear form, as is common in the field of industrial organization. I will assume that the demand side of the industry can be modeled by a representative consumer with quadratic utility function:
U(q) = v \frac{N}{2(1+\mu)} \left[ \sum_{i=1}^{N} q_i^2 + \frac{\mu}{N} \left( \sum_{i=1}^{N} q_i \right)^2 \right]

where \( I \) is the numeraire good, \( N \) is the number of firms/products and \( \mu \in [0, \infty) \) represents the level of symmetric product differentiation. This formulation of the utility function dates back to Levitan & Shubik (1971), and is used in several textbooks, such as Vives (2001) and Motta (2004). Marshallian demand for product \( i \) is:

\[ q_i = \frac{1}{N} \left[ v - \mu \left( p_i - \frac{1}{N} \sum_{j=1}^{N} p_j \right) \right] \]

Demand for product \( i \) directly decreases in firm \( i \)'s own price. Additionally, demand is indirectly affected by how much firm \( i \)'s price differs from the average price in the industry. If \( p_i \) is below the average market price, demand for firm \( i \)'s product increases. Conversely, if \( p_i \) is above the average market price, demand for firm \( i \)'s product decreases. How large this indirect effect is, depends on the level of product differentiation. The more differentiated the products are (the smaller \( \mu \)), the smaller is the indirect effect. Another attractive feature of this formulation of the demand functions, as Motta (2004) points out, is that aggregate demand is independent of the number of firms and the substitutability between their products.

2.2 Modelling firms

Let \( \mathbf{N} \) denote the set of firms in a market. Furthermore, let \( \kappa \subseteq \mathbf{N} \) denote the set of cartel members in the market. Assume that costs are linear and that cartel-members set prices to maximize the cartel’s joint profit:

\[ \max_{\{p_i\}_{i \in \kappa}} \sum_{j \in \kappa} (p_j - c) q_j(p_j, p_{-j}) \]

Assume that non-members set prices to maximize their own profit. Thus, for each \( i \notin \kappa \) the optimization problem is:

\[ \max_{p_i} (p_i - c) q_i(p_i, p_{-i}) \]

For simplicity, assume that firms have symmetric marginal costs that are equal to zero. Namely, \( c_i = c_j = 0 \) for all \( i, j \in \mathbf{N} \).
3 Equilibrium of the repeated price game

3.1 Equilibrium prices and profits

Let $K$ and $N$ denote the number of cartel members and firms in a market, respectively. As firms are symmetric, I find $K$ identical first-order conditions for cartel members and $(N-K)$ identical first-order conditions for non-members. In equilibrium, there will consequently be two sets of prices: the cartel price and the non-members’ price, denoted by $p_{κ}(K,N,μ)$ and $p_{NM}(K,N,μ)$, respectively:

$$p_{NM}(K,N,μ) = \frac{Nv((2N-K)μ + 2N)}{(K+2N-2)(N-K)μ^2 + 2N(3N-K-1)μ + 4N^2}$$

(5)

$$p_{κ}(K,N,μ) = \frac{Nv((2N-1)μ + 2N)}{(K+2N-2)(N-K)μ^2 + 2N(3N-K-1)μ + 4N^2}$$

(6)

Thus, equilibrium profits are also functions of cartel size $K$, market size $N$ and product homogeneity $μ$:

$$π_{NM}(K,N,μ) = \frac{v^2((N-1)μ + N)((2N-K)μ + 2N)^2}{((K+2N-2)(N-K)μ^2 + 2N(3N-K-1)μ + 4N^2)^2}$$

(7)

$$π_{κ}(K,N,μ) = \frac{v^2((N-K)μ + N)((2N-1)μ + 2N)^2}{((K+2N-2)(N-K)μ^2 + 2N(3N-K-1)μ + 4N^2)^2}$$

(8)

As Deneckere & Davidson (1985) showed, profits are increasing in the size of a coalition, and free-riders benefit more than coalition members. This is illustrated numerically in Figures 2 and 3. The model also illustrates the intuitive result that the damage caused by the cartel is increasing in the cartel share. Furthermore, the marginal damage of an increase in cartel share is also increasing (as the profit functions are convex in the cartel share).

When product homogeneity $μ$ increases, the cartel profit $π_{κ}$ is reduced for all cartel sizes $K < N$. Note that $π_{κ}(N,N,μ) = \frac{v^2}{4N}$. Thus, the cartel-profit with a market-wide cartel is independent of $μ$. The Nash equilibrium profits $π_{κ}(1,N,μ) = π_{NM}(1,N,μ) = π_{NE}(N,μ)$ are strictly decreasing in $μ$. Numerically, I find that the non-members’ profits are increasing (decreasing) in $μ$ when the cartel is sufficiently large (small).

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3 For more detailed calculation of equilibrium prices and profits, see appendix A.

4 Note that there is a difference in scaling of the two figures.

5 Note that $π_{NE} = \frac{v^2(N+(N-1)μ)}{(2N+(N-1)μ)}$ and $\frac{∂π_{κ}}{∂μ} = -\frac{v^2(N-1)μ}{(2N+(N-1)μ)}$. 
3.2 Incentive compatibility

For a cartel to be sustainable, the pay-off from colluding must be larger than the pay-off from deviating. As previously, let \( \pi_\kappa(K, N, \mu) \) denote a cartel member’s profit when colluding. Note that the pay-off in Nash-equilibrium is \( \pi_{NE}(N, \mu) = \pi_\kappa(K, N, \mu) |_{K=1} \). Furthermore, let \( \pi_D(K, N, \mu) \) denote the firm’s profit when deviating from the collusive agreement.\(^6\) As I assume that cartel breakdown leads to reversion to the Nash-equilibrium forever (grim-trigger strategies), the incentive compatibility constraint (ICC) can be expresses as:

\[
\frac{\pi_\kappa(K, N, \mu)}{1 - \delta} \geq \pi_D(K, N, \mu) + \frac{\delta}{1 - \delta} \pi_{NE}(N, \mu),
\]

where \( \delta \) denotes the firms’ common discount factor, which can be interpreted as the firms’ patience. Solving for \( \delta \), the ICC identifies firms’ critical level of patience, \( \delta_{min} \), necessary to sustain the collusive arrangement:

\[
\delta \geq \frac{\pi_D(K) - \pi_\kappa(K)}{\pi_D(K) - \pi_{NE}} \equiv \delta_{min}
\]

A numerical illustration of the critical discount factor for a given market size is given in figure 4. The figure shows that critical patience is decreasing (increasing) in product homogeneity \( \mu \) for sufficiently small (large) cartels. Furthermore, critical patience is strictly increasing in cartel size \( K \).

It is not immediately clear that critical patience should increase with the cartel size. On

\(^6\)For more detailed derivation of \( \pi_D(K) \), I refer to appendix B.
the left-hand side of the ICC, a larger cartel leads to a higher cartel profit. This suggests that the ICC should be easier to satisfy, and that $\delta_{\text{min}}$ should decrease. However, a larger cartel also makes deviations more profitable, as more firms set the high cartel price. As prices are strategic complements, also the non-members set higher prices. When deviating, a firm can therefore capture a larger demand, thus making a higher profit. The increase in deviation-profit leads to a more stricter ICC, and consequently a larger $\delta_{\text{min}}$. In this model, the latter effect dominates, implying that larger cartels require a higher level of patience:

**Proposition 1** Let the critical discount factor be defined by (10), then for all $2 \leq N$ and $2 \leq K \leq N$:

$$\frac{\partial \delta_{\text{min}}}{\partial K} \geq 0$$  \hspace{1cm} (11)

Proof of the proposition is provided in appendix B.

Of further interest is how the ICC is affected by product differentiation. Intuitively, if the firms’ products are closer substitutes, the free-riders can gain more by joining the cartel, internalizing a larger externality. However, non-members also benefit more from free-riding, as the cartel members’ increased prices lead to a larger increase in the free-rider’s demand. It is therefore not clear whether increased differentiation makes the cartel more or less inclusive. According to proposition 1, the ICC is strictest when the cartel is market-wide. When determining the maximum patience required for the ICC to hold, it therefore suffices to assess the ICC for market-wide cartels:

**Proposition 2** Consider a market-wide cartel, such that $K = N$. Then the critical discount
The factor defined in (10) is increasing in the level of product homogeneity:

$$\frac{\partial \delta_{\min}}{\partial \mu} \bigg|_{K=N} \geq 0$$

(12)

And $\delta_{\min}$ reaches its maximum when $\mu$ tends to infinity:

$$\lim_{\mu \to \infty} \delta_{\min} \bigg|_{K=N} = 1$$

(13)

A proof is provided in appendix B.

The results imply that if firms are sufficiently patient ($\delta$ is sufficiently close to 1), the ICC holds. In reality, discount factors tend to be quite large (typically in the range 0.95 – 0.99). As $\frac{\partial \delta_{\min}}{\partial K} \geq 0$, this implies that partial cartels are likely to be incentive compatible. The numerical results suggest that the ICC is not likely to be violated when product differentiation is sufficiently large ($\mu$ is sufficiently low). This leads to the conclusion that large cartels may not be incentive compatible when products are sufficiently homogeneous. Otherwise the ICC is likely to hold.

4 Equilibrium of the cartel formation stage

The concepts of internal and external cartel stability was first introduced by d’Aspremont et al. (1983). A cartel is considered internally stable if no participant strictly prefers to leave the cartel. Similarly, a cartel is considered externally stable if no free-rider strictly prefers to join the cartel. Given the firms’ profit functions in (7) and (8), I define cartel stability as follows:

**Definition 1** Let $\Pi_k(K)$ and $\Pi_{NM}(K)$ denote the profits of cartel members and non-members, respectively. A cartel of size $K$ is:

- **Internally stable** iff $\pi_k(K) \geq \pi_{NM}(K-1)$

- **Externally stable** iff $\pi_{NM}(K) \geq \pi_k(K+1)$

Figures 5 and 6 correspond to figures 2 and 3, replacing the graph for $\pi_{NM}(K)$ with that of $\pi_{NM}(K-1)$. Note that the difference between the figures is simply that the graph for $\pi_{NM}(K)$ has shifted to the right. Figure 5 illustrates an example with a unique stable cartel, identified by the point where the two graphs cross. Denote this value of $K$ by $K^*$. Thus, the stable cartel size $\hat{K}$ will be the largest integer weakly below $K^*$. Note that $\pi_k(K^*) = \pi_{NM}(K^* - 1)$ only identifies a stable cartel when $\pi_{NM}(K-1)$ crosses $\pi_k(K)$ from below. Namely, it must be the case that $\frac{\partial \pi_k(K^*)}{\partial K} < \frac{\partial \pi_{NM}(K^* - 1)}{\partial K}$. In such a solution, the cartel is both internally and externally stable.

Through implicit differentiation, I can assess changes in $K^*$, when the level of product homogeneity increases. Intuitively, when competition ($\mu$) increases, the firms lose profit. When
non-members lose more than cartel members, cartel membership becomes more attractive. The cartel size consequently increases. This is shown mathematically as follows:

\[
\begin{align*}
\left[ \frac{\partial \pi_{\text{NM}}(K^* - 1)}{\partial K} - \frac{\partial \pi_{\kappa}(K^*)}{\partial K} \right] dK^* = & \left[ \frac{\partial \pi_{\kappa}(K^*)}{\partial \mu} - \frac{\partial \pi_{\text{NM}}(K^* - 1)}{\partial \mu} \right] \\
\downarrow & \downarrow \\
\text{sign} \left[ \frac{dK^*}{d\mu} \right] = & \text{sign} \left[ \frac{\partial \pi_{\kappa}(K^*)}{\partial \mu} - \frac{\partial \pi_{\text{NM}}(K^* - 1)}{\partial \mu} \right]
\end{align*}
\]

If product homogeneity \( \mu \) is sufficiently large, the two graphs do not cross. Consequently, \( \pi_{\kappa}(K) \geq \pi_{\text{NM}}(K - 1) \) holds for all \( K \leq N \). This implies that the external stability condition is violated if \( K < N \). The unique stable cartel is therefore the corner solution \( \hat{K} = N \). This is illustrated in figure 6.

Figure 6 suggests that there is an upper limit for homogeneity \( \mu_{\text{cut-off}} \), such that \( \hat{K} = N \) for any \( \mu \geq \mu_{\text{cut-off}} \). In other words, when products are sufficiently homogeneous, the market-wide cartel is a stable solution. In fact, for any market size \( N \), I can find a the value of \( \mu_{\text{cut-off}} \). These values are illustrated in figure 7. As the figure shows, the cut-off value for stability of the market-wide cartel are increasing in the number of firms in the industry.

The values of \( \mu_{\text{cut-off}} \) may be little intuitive, as it is difficult to determine what constitutes a "large" or "small" level of homogeneity. It may therefore be useful to consider so-called diversion ratios. Denote the diversion from firm \( i \) to firm \( j \) as \( D_{ij} \). Then the diversion ratio is
Figure 6: Profits for cartel members and non-members, $N = 7, v = 1$ and $\mu = 20$

Figure 7: Values for $\mu_{\text{cut-off}}$ as a function of $N$ given by

$$D_{ij} \equiv -\frac{\partial q_j}{\partial p_i}.$$  \hspace{1cm} (14)

The diversion ratio has a more intuitive interpretation than $\mu$. When firm $i$ raises its price, it loses some demand, $-\frac{\partial q_j}{\partial p_i}$. However, firm $j$ gains demand due to its competitor’s price increase,
Thus, the diversion ratio is the increase in firm $j$’s demand as a share of firm $i$’s loss. For example, a diversion ratio of $D_{ij} = 0.2$ implies that if firm $i$ raises its price, firm $j$ is able to capture 20 percent of the demand firm $i$ has lost. For Shubik-Levitan preferences and market-wide cartels, the symmetric diversion ratios are:

$$D_{ij}(N, \mu) = \frac{\mu}{N + (N - 1)\mu}$$

(15)

I can now assess the diversion ratios at $\mu_{\text{cut-off}}$. As the diversion ratios are increasing in $\mu$, diversion must be above $D_{ij}(N, \mu_{\text{cut-off}})$ for the market-wide cartel to be stable. A relevant comparison to $D_{ij}(N, \mu_{\text{cut-off}})$ is the maximum possible level of diversion:

$$\lim_{\mu \to \infty} D_{ij}(N, \mu) = \frac{1}{N - 1}$$

(16)

Maximum diversion thus implies that aggregate demand is fixed (i.e. consumers do not use the outside option). Thus, when firm $i$ raises its price, the change in its demand is distributed evenly among the $(N - 1)$ remaining firms in the industry. Diversion at the cut-off value for $\mu$ and maximum diversion are illustrated in figure 8. As the figure illustrates, cut-off diversion for the market-wide cartel converges to maximum diversion as $N$ increases. Furthermore, convergence appears to occur relatively fast. This implies that stability of a market-wide cartel requires relatively high levels of product homogeneity. Furthermore, market-wide cartels are most likely to occur in small industries, where internal stability is satisfied also for relatively differentiated products.

![Figure 8: Diversion at $\mu_{\text{cut-off}}$ and maximum diversion as functions of $N$](image)

For a given market size, I can find a cut-off value for $\mu$ such that a cartel of size $K$ is
internally stable. For a market with $N = 7$ firms, these cut-off values are illustrated in figure 9. For a given level of product homogeneity $\mu$, I am now able to find the stable cartel in this market. For example, if $\mu = 8$ all cartels of size $K \leq 5$ are internally stable, as $\mu_{\text{cut-off}}(K) < 8$. However, larger cartels are not internally stable. Consequently, the unique stable cartel size is $\hat{K} = 5$, as it satisfies both internal and external stability. If competition were to become tougher, namely if $\mu$ increases, the cartel size will weakly increase as well. For example, if $\mu$ increases to $\mu = 10$, the stable cartel would now be of size six, $\hat{K} = 6$. This is illustrated in figure 10.

Figure 9: Cut-off values for $\mu$, such that a cartel of size $K$ is internally stable, $N = 7$

Figure 10: Cut-off values for $\mu$, such that a cartel of size $K$ is internally stable, $N = 7$

Numerically, I find that the cut-off values for $\mu$ are non-monotonic in $K$ when markets are sufficiently large (i.e. when $N \geq 8$). For small levels of homogeneity, there still exists a unique stable cartel size $\hat{K}$. As $\mu$ increases, there may be two stable cartels – a partial cartel and the market-wide cartel. This is illustrated in figures 11 and 12.
Figure 11: Cut-off values for $\mu$, such that a cartel of size $K$ is internally stable, $N = 9$

Figure 12: Cut-off values for $\mu$, such that a cartel of size $K$ is internally stable, $N = 9$

Figure 13 illustrates why multiple solutions occur. For parameter values $N = 9$ and $\mu = 30$, the profit functions from the internal stability condition intersect twice: once between $K = 6$ and $K = 7$ and a second time between $K = 8$ and $K = 9$. Both intersections are (weakly) internally stable. However, only the first intersection is externally stable as well. Denote the point of the second intersection as $K_2^*$. Then it is easy to see that $\pi_K(K_2^*) > \pi_{NM}(K_2^*)$. Thus, external stability is violated in $K = K_2^*$. Finally, the corner solution is also a stable solution as the internal stability condition holds when $K = N$.

Figure 13: Multiple stable cartels, $N = 9$ and $\mu = 30$
Figures 14-17 illustrate the maximum size for a stable cartel for markets of different sizes. When \( \mu > \mu_{\text{cut-off}}|_{K=N} \), the maximum stable cartel is market-wide. When \( \mu < \mu_{\text{cut-off}}|_{K=N} \) the optimal cartel size \( K^* \) is derived from the internal stability. Namely, \( K^* \) is given implicitly by \( \pi_K(K^*) = \pi_{NM}(K^* - 1) \). The actual cartel size of the stable cartel is thus the largest integer \( \hat{K} \leq K^* \). The figures suggest that partial cartels are of limited size, irrespective of the size of the industry. Furthermore, I find that the size of the largest stable partial cartel is \( \hat{K} = 6 \) if \( N > 10 \).

Figure 14: Maximum stable cartel size, \( N = 6 \)

Figure 15: Maximum stable cartel size, \( N = 9 \)

Figure 16: Maximum stable cartel size, \( N = 15 \)

Figure 17: Maximum stable cartel size, \( N = 75 \)
5 Conclusions and further research

In a model with symmetric firms, linear costs and quadratic profit functions, I find that the size of a (stable) cartel decreases (increases) when products become more differentiated (homogeneous). Furthermore, I find that the size of the smallest stable cartel rarely exceeds six to seven firms, as the relative pay-off from free-riding increases faster in cartel size than the pay-off from collusion – irrespective of industry size. Finally, I find that industry-wide cartels may not be incentive compatible when products are sufficiently homogeneous. Otherwise, the incentive compatibility constraint likely holds.

In this draft, I have presented numerical solutions to the cartel-formation stage. Much work remains to derive corresponding analytical results, and computations need to be proofread. This will be the main focus of my continued work with the model. Subsequently, I may consider extensions to the model. Introducing firm heterogeneity may be one such extension. Introducing heterogeneous product differentiation may yield important insights into coordinated effects of merger. However, this may be intractable analytically. Another possible extension is to allow for renegotiation after cartel break down.
References


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A Calculation of equilibrium prices and profits

A.1 Cartel members’ first-order conditions

The cartel maximizes its joint profit, subject to the cartel members’ prices, according to (3). The maximization problem yields the following first-order condition with respect to the price of cartel member $i$:

$$\sum_{j \neq i, j \in \kappa} p_j \frac{\partial q_j}{\partial p_i} + p_i \frac{\partial q_i}{\partial p_i} + q_i = 0,$$

where demand $q_i, q_j$ is given by (2). Inserting for the cross- and own-price derivatives, and using that firms are symmetric, I find:

$$(K - 1) \frac{\mu}{N^2} p_{\kappa} - \frac{N + (N - 1) \mu}{N^2} N^2 p_{\kappa} + \frac{v}{N} - \frac{N + (N - K) \mu}{N^2} N^2 p_{\kappa} + (N - K) \mu \frac{v}{N^2} = 0,$$

where $K$ is the number of firms in the cartel, $N$ is the number of firms in the industry, and $\mu$ is the parameter for product homogeneity for the demand function. Rewriting yields:

$$(N - K) \mu p_{NM} - 2(N + (N - K) \mu) p_{\kappa} = -Nv$$  \hspace{1cm} (17)

A.2 Non-members’ first-order conditions

The non-members set their price to maximize their own profit, as given in (4). The first-order condition for non-member $i$ is:

$$p_i \frac{\partial q_i}{\partial p_i} + q_i = 0$$

Substituting for the own-price derivative, and using that firms are symmetric, I find:

$$-(2N + (N - 1 + K) \mu) p_{NM} + K \mu p_{\kappa} = -Nv$$  \hspace{1cm} (18)
A.3 Equilibrium prices, demand and profits

Combining the first-order conditions for non-members and cartel members in (17) and (18), I find equilibrium prices as function of cartel size $K$, market size $N$ and product homogeneity $\mu$:

\[
p_{NM}(K, N, \mu) = \frac{Nv((2N-K)\mu + 2N)}{(K+2N-2)(N-K)\mu^2 + 2N(3N-K-1)\mu + 4N^2}
\]

\[
p_{K}(K, N, \mu) = \frac{Nv((2N-1)\mu + 2N)}{(K+2N-2)(N-K)\mu^2 + 2N(3N-K-1)\mu + 4N^2}
\]

Furthermore, I find the demand for non-members and cartel members by substituting the equilibrium prices into the demand functions:

\[
q_{NM}(K, N, \mu) = \frac{v}{N} - \frac{N + K\mu}{N^2} p_{NM}(K, N, \mu) + \frac{K}{N^2} p_{K}(K, N, \mu)
\]

\[
q_{K}(K, N, \mu) = \frac{v}{N} - \frac{N + (N-K)\mu}{N^2} p_{K}(K, N, \mu) + \frac{(N-K)\mu}{N^2} p_{NM}(K, N, \mu)
\]

Thus, equilibrium profits are also functions of cartel size $K$, market size $N$ and product homogeneity $\mu$:

\[
\pi_{NM}(K, N, \mu) = p_{NM}(K, N, \mu) q_{NM}(K, N, \mu) = \frac{v^2((N-1)\mu + N)((2N-K)\mu + 2N)^2}{((K+2N-2)(N-K)\mu^2 + 2N(3N-K-1)\mu + 4N^2)^2}
\]

\[
\pi_{K}(K, N, \mu) = p_{K}(K, N, \mu) q_{K}(K, N, \mu) = \frac{v^2((N-K)\mu + N)((2N-1)\mu + 2N)^2}{((K+2N-2)(N-K)\mu^2 + 2N(3N-K-1)\mu + 4N^2)^2}
\]

B The effect of parameter values on the ICC

Let the profit from collusion, Nash-equilibrium and deviation be

\[
\pi_{K}(K) = \frac{v^2((N-K)\mu + N)((2N-1)\mu + 2N)^2}{((K+2N-2)(N-K)\mu^2 + 2N(3N-K-1)\mu + 4N^2)^2}
\]

\[
\pi_{NE} = \pi_{K}(1) = \frac{v^2(N+\mu(N-1))}{(2N+\mu(N-1))^2}
\]

\[
\pi_{D}(K) = \frac{v^2((K-2N+1)\mu - 2N)^2((N-\frac{1}{2})\mu + N)^2}{((K+2N-2)(K-N)\mu^2 + 2N(K-3N+1)\mu - 4N^2)^2(N+\mu(N-1))}
\]
respectively. \( \pi_{\kappa}(K) \) and \( \pi_{NE} \) are equilibrium cartel profits, derived from the market equilibrium. The profit from deviation, \( \pi_{D} \), is derived as follows. A deviating cartel-member sets a price \( p_{D} \) to maximize its profit. However, other firms in the market are unable to respond to the unanticipated deviation. This yields the first-order condition:

\[
p_{D}(K) \frac{\partial q_{i}}{\partial p_{i}} + q_{D}(K) = 0,
\]

where \( q_{D}(K) = \frac{1}{N} \left[ \nu - p_{D}(K) - \mu \left( p_{D}(K) - \frac{(N-K)p_{NM}(K)+Kp_{\kappa}(K)}{N} \right) \right]. \]

The critical level of patience, \( \delta_{\text{min}} \), required for the incentive compatibility constraint (ICC) to be satisfied, is given by:

\[
\delta_{\text{min}} = \frac{\pi_{D}(K) - \pi_{\kappa}(K)}{\pi_{D}(K) - \pi_{NE}}, \tag{28}
\]

Thus, the critical level of patience can be expressed as:

\[
\delta_{\text{min}} = \frac{\left(2N + \mu(N-1)\right)^2}{\left(K - \frac{1}{2}\right)(N-1)\mu + N(K-1)\left(16N^2 - (2K + 4)N - K^2 + \frac{5}{2}K + \frac{1}{2}\right)\mu^2 + N\left(16N^2 - (6K + 20)N - K^2 + 6K + 5\right)\mu^2 + 4N^2(5N - K + 3)\mu + 8N^3}, \tag{29}
\]

**B.1 The change in critical patience with respect to cartel size \( K \)**

Differentiating (29) with respect to \( K \) yields:

\[
\frac{\partial \delta_{\text{min}}}{\partial K} = \frac{2\left(2N + \mu(N-1)\right)^2}{\left(K - \frac{1}{2}\right)(N-1)\mu + N(K-1)\left(16N^2 - (2K + 4)N - K^2 + \frac{5}{2}K + \frac{1}{2}\right)\mu^2 + N\left(16N^2 - (6K + 20)N - K^2 + 6K + 5\right)\mu^2 + 4N^2(5N - K + 3)\mu + 8N^3}, \tag{30}
\]

The denominator of this fraction consists of squared terms, and must therefore be positive. As \( K \) and \( N \) are positive integers, with relevant range \( N \geq 2 \) and \( 2 \leq K \leq N \), it must be the case that:

\[
\text{sign} \left[ \frac{\partial \delta_{\text{min}}}{\partial K} \right] = \text{sign} \left[ (N^2 + (K^2 - 2K - \frac{1}{2})N + K^3 - 3K^2 + 3K - \frac{1}{2})(N-1)\mu^2 + N\left(3N^2 + (3K^2 - 6K - 2)N + K^3 - 5K^2 + 7K - 1\right)\mu + 2N^2(K^2 - 2K + N) \right]
\]

Assessing the terms separately, if begin by checking when

\[
\left( N^2 + (K^2 - 2K - \frac{1}{2})N + K^3 - 3K^2 + 3K - \frac{1}{2} \right)(N-1)\mu^2 \geq 0
\]

Finding the derivatives with respect to \( N \) and \( K \) yields:

\[
\frac{\partial}{\partial N} \left( N^2 + (K^2 - 2K - \frac{1}{2})N + K^3 - 3K^2 + 3K - \frac{1}{2} \right) \geq 0 \quad \text{for all} \quad N \geq 2, 2 \leq K \leq N
\]

\[
\frac{\partial}{\partial K} \left( N^2 + (K^2 - 2K - \frac{1}{2})N + K^3 - 3K^2 + 3K - \frac{1}{2} \right) \geq 0 \quad \text{for all} \quad N \geq 2, 2 \leq K \leq N
\]
Thus, I need only assess the term in \((N, K) = (2, 2)\):

\[
\left. \left( N^2 + (K^2 - 2K - \frac{1}{2})N + K^3 - 3K^2 + 3K - \frac{1}{2} \right) \right|_{(N, K) = (2, 2)} = \frac{9}{2} > 0
\]

Thus, it must be the case that:

\[
\left( N^2 + (K^2 - 2K - \frac{1}{2})N + K^3 - 3K^2 + 3K - \frac{1}{2} \right)(N - 1)\mu^2 \geq 0 \quad \text{for all } N \geq 2, 2 \leq K \leq N
\]

Similarly, I can show that

\[
N\left( 3N^2 + (3K^2 - 6K - 2)N + K^3 - 5K^2 + 7K - 1 \right)\mu \geq 0 \quad \text{for all } N \geq 2, 2 \leq K \leq N,
\]

as:

\[
\frac{\partial}{\partial N}\left( 3N^2 + (3K^2 - 6K - 2)N + K^3 - 5K^2 + 7K - 1 \right) \geq 0 \quad \text{for all } N \geq 2, 2 \leq K \leq N
\]

\[
\frac{\partial}{\partial K}\left( 3N^2 + (3K^2 - 6K - 2)N + K^3 - 5K^2 + 7K - 1 \right) \geq 0 \quad \text{for all } N \geq 2, 2 \leq K \leq N
\]

\[
\left. \left( 3N^2 + (3K^2 - 6K - 2)N + K^3 - 5K^2 + 7K - 1 \right) \right|_{(N, K) = (2, 2)} = 2 > 0
\]

I therefore conclude that the numerator must also be positive for the relevant range of parameter values. This implies that:

\[
\frac{\partial \delta_{\text{min}}}{\partial K} \geq 0 \quad (31)
\]

Consequently, the largest values of required patience, \(\delta_{\text{min}}\), occur when the cartel is market wide (i.e. when \(K = N\)).

B.2 The change in critical patience with respect to \(\mu\), when \(K = N\)

As \(\frac{\partial \delta_{\text{min}}}{\partial K} \geq 0\), the critical level of patience occurs when the cartel is market wide. It is therefore sufficient to assess the incentive compatibility constraint (ICC) when \(K = N\). If the ICC holds for the market-wide cartel, it will also hold for any partial cartel with \(K < N\). Substituting for \(K = N\) in (29) yields:

\[
\delta_{\text{min}}(K = N) = \frac{(N - 1)^2 \mu + 4N(N - 1)\mu + 4N^2}{(N - 1)^2 \mu + 8N(N - 1)\mu + 8N^2} \quad (32)
\]
Assessing patience in the limits of $\mu$:

\[
\lim_{\mu \to 0} \delta_{\min}(K = N) = \frac{1}{2} \\
\lim_{\mu \to \infty} \delta_{\min}(K = N) = 1
\]

In fact, $\lim_{\mu \to 0} \delta_{\min} = \frac{1}{2}$ for all $K \leq N$. Furthermore, the critical level of patience is strictly increasing in $\mu$ when $K = N$:

\[
\frac{\partial \delta_{\min}(K = N)}{\partial \mu} = \frac{4(N - 1)^2 N^2}{\left(9N^2 \mu + 8N^2 - 10N \mu + \mu\right)^2} \geq 0 \quad (33)
\]

Thus, if firms are sufficiently patient, the incentive compatibility constraint is satisfied. As the critical level of patience is increasing in $K$, the ICC is likely satisfied when products are differentiated and/or cartels are partial. However, if products are sufficiently homogeneous, a market-wide cartel may not be incentive compatible.

C A note on microeconomic foundations

Firstly, the utility function must be concave in each good separately. Differentiating (1) with respect to $q_i$, we find that:

\[
\frac{\partial U}{\partial q_i} = v - \frac{N + \mu}{1 + \mu} q_i - \frac{\mu}{1 + \mu} \sum_{j \neq i} q_j \quad (34)
\]

Thus, the utility function is strictly concave in each good separably if $N > 0$ for all values of $\mu$ in its relevant range (recall that $\mu \in [0, \infty)$).

Secondly, Amir et al. (2017) show that it is necessary for a quadratic utility function to be strictly concave, for the demand system to be well defined. Define $\theta \equiv \frac{N + \mu}{1 + \mu}$ and $\gamma \equiv \frac{\mu}{1 + \mu}$. Then the price vector $p$ is given by the first order conditions from the utility maximization problem:

\[
p = v - Bq, \quad (35)
\]

where $v$ is the vector containing the scalar $v$ in each entry, $q$ is the vector of quantities and the matrix $B$ is given by:

\[
B = \begin{bmatrix}
\theta & \gamma & \ldots & \gamma \\
\gamma & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \gamma \\
\gamma & \ldots & \gamma & \theta
\end{bmatrix} \quad (36)
\]

Amir et al. (2017) show that if the matrix $B$ is positive semi-definite (the quadratic utility
function is concave), then the demand functions given in (2) satisfies the Law of Demand. Following the methodology of Amir et al. (2017), I rewrite $\mathbf{B}$ as:

$$\mathbf{B} = \gamma \mathbf{J}_N + (\theta - \gamma) \mathbf{I}_N,$$

where $\mathbf{J}_N$ is the $N \times N$ matrix of 1s and $\mathbf{I}_N$ is the $N \times N$ identity matrix. To find the eigenvalues of $\mathbf{B}$, consider the matrix $\mathbf{B} - \lambda \mathbf{I}_N$. The determinant of this matrix can be found using the matrix determinant lemma:

$$\det[\mathbf{B} - \lambda \mathbf{I}_N] = \det[\gamma \mathbf{J}_N + (\theta - \gamma - \lambda) \mathbf{I}_N] = (\theta - \gamma - \lambda)^{N-1}\left[\theta - \lambda + (N - 1)\gamma\right]$$

Thus, the determinant $\det[\mathbf{B} - \lambda \mathbf{I}_N] = 0$ if $\lambda = \theta - \gamma$ or $\lambda = \theta + (N - 1)\gamma$. It is trivial to show that $\lambda > 0$ for the entire range of $\mu \in [0, \infty)$ if $N > 0$. Consequently, the utility function in (1) is strictly concave, and the linear demand functions in (2) satisfy the Law of Demand.